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Universal Norms on Abelian Varieties over Global Function Fields

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We examine the Mazur–Tate canonical height pairing defined between an abelian variety over a global field and its dual. We show in the case of global function fields

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1. INTRODUCTION

In [10], Mazur and Tate determined general methods for constructing canonical height pairings

$$A(K) \times A'(K) \to Y, \tag{1}$$

where A is an abelian variety over a global field K, A' is its dual, and Y is an abelian group. Their methods built on the work of Bloch [2] and Schneider [19] and involved canonical splittings of the Poincaré biextension of A.

Our goal in this paper is to analyze the Mazur–Tate construction of certain height pairings defined for abelian varieties over global fields, and in particular over global function fields. We begin in Sections 3 and 4 by outlining the definition of the Mazur–Tate pairing in (1) and its decomposition into local terms.

In Section 5 we investigate the nature in which certain of these local heights can be obtained from nonarchimedean theta functions. Here the theta functions we use were defined by Norman [16, 17]; in general such

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theta functions have been studied extensively in other contexts (see Barsotti [1], Breen [3], Candilera and Cristante [4], Cristante [5, 6], and Turner [20]). We pay special attention to the field of definition and do not restrict to an algebraically closed ground field.

Of note we show that if A is an elliptic curve, then Norman's theta function coincides with the Mazur–Tate sigma function of [11].

In Section 6 we specialize to the case that $K/\mathbb{F}_q(t)$ is a global function field in positive characteristic $p \geqslant 3$. If v is a finite place of $k = \mathbb{F}_q(t)$ and A/K has good ordinary reduction at each place of K extending v, then there is a natural Mazur–Tate pairing

$$\langle\ ,\ \rangle_v \colon A(K) \times A'(K) \to \mathbb{C}_v^\times,$$

where $\mathbb{C}_v$ is the completion of the algebraic closure of k_v.

We show that this pairing is annihilated by the universal norms in $A(K)$ coming from the Carlitz cyclotomic extension which is totally ramified at v and unramified away from v. This can be seen as an analogy with global p-adic heights; for abelian varieties over number fields Schneider shows that p-adic heights are trivial on universal norms coming from certain $\mathbb{Z}_p$-extensions (see [10, Sect. 4]).

Using the results of Section 5, we find that for elliptic curves this Mazur–Tate height coincides with the heights defined in [18]. We show that when A is an elliptic curve these universal norm groups are trivial in certain cases.

2. PRELIMINARIES

We will use the following conventions. We let K denote a global field, in other words a number field or function field in one variable over a finite field, and let R be its ring of integers. For a place w of K, we let K_w and R_w be the completions at w and $\mathbb{F}_w$ the residue field. For simplicity we will assume that $\mathbb{F}_w$ has characteristic $p \geqslant 3$, although we make no assumptions on the characteristic of K_w until Section 6, where K_w will have equal characteristic. We let F denote a general field.

Let A be an abelian variety of dimension d defined over a field F and take A' for the dual abelian variety of A. For any integer n, $[n] \colon A \to A$ is the multiplication by n map and $A[n]$ is its kernel.

We let $\mathrm{Div}(A)$ denote the group of divisors on A which are defined over F^{sep} and take $\mathrm{Div}_F(A)$ for the subgroup of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-invariant divisors. For two divisors C and D on A, we write $C \sim D$ if they are linearly equivalent and $C \approx D$ if algebraically equivalent. Let $\mathrm{Pic}^0(A)$ be the group of divisors algebraically equivalent to 0 modulo linear equivalence.

The zero cycles of A comprise the group ring $\mathbb{Z}[A(\bar{F})]$. For $\alpha \in A(\bar{F})$ we write (α) for its image in $\mathbb{Z}[A(\bar{F})]$. The augmentation ideal $I[A]$, or rather the zero cycles of degree zero, is the kernel of the degree map $\mathbb{Z}[A(\bar{F})] \to \mathbb{Z}$. The kernel of the natural summation map, $I[A] \to A(\bar{F})$, is the square ideal $I[A]^2$.

For f a rational function in $\bar{F}(A)$ and $\mathfrak{a} = \sum m_\alpha(\alpha)$ in $I[A]$ with disjoint supports, we take

$$f(\mathfrak{a}) := \prod_{\alpha} f(\alpha)^{m_\alpha} \in \bar{F}^\times,$$

the value of which depends only on $\operatorname{div}(f)$ and $\mathfrak{a}$.

For a divisor D and a point $\alpha \in A(\bar{F})$, the translation of D by α, the divisor denoted D_α, is the image of D under the translation map $\tau_\alpha \colon \beta \mapsto \beta + \alpha$. Thus $D_\alpha = \tau_{-\alpha}^* D$. Likewise, we define the translation $\mathfrak{a}_\alpha$ for any $\mathfrak{a} \in \mathbb{Z}[A(\bar{F})]$. For any divisor $D \in \operatorname{Div}(A)$, we let $\phi_D \colon A \to A'$ be the polarization map which sends $\alpha \in A(\bar{F})$ to the class of $D - \tau_\alpha^* D = D - D_{-\alpha}$ in $\operatorname{Pic}^0(A)$.

Now suppose that $F = K_w$ is a local field. Let $\mathscr{A}$ be the Néron model of A over R_w with special fiber $\mathscr{A}_0$, and let $\mathscr{A}^0$ be its connected component of the identity. We take $\hat{A}$ to be the formal group of $\mathscr{A}$ over R_w, i.e., the formal completion of $\mathscr{A}$ along its zero section. For the valuation ring $\bar{R}_w$ in some fixed algebraic closure $\bar{K}_w$, the zero cycles $\mathbb{Z}[\hat{A}(\bar{R}_w)]$ of $\hat{A}$ and its augmentation ideal $I[\hat{A}]$ are defined as above.

3. BIEXTENSIONS AND ρ-SPLITTINGS

If F is any field and A is defined over F as in the previous section, let E be the Poincaré biextension $E \to A \times A'$ of (A, A') by $\mathbb{G}_m$, which provides the duality between A and A'. We note that $E(F)$ is in fact a set-theoretic biextension of $(A(F), A'(F))$ by $F^\times$. If $F = K_w$ is a local field, then we further define $\mathscr{E}$ to be the canonical biextension of $(\mathscr{A}^0, \mathscr{A}')$ by $\mathbb{G}_m/R_w$, i.e., the unique such biextension which has E as its generic fiber. Furthermore, we let $\hat{E}$ be the formal biextension associated to $\mathscr{E}$, which is the formal completion of $\mathscr{E}$ along the inverse image of the zero section of $\mathscr{A}^0 \times_{R_w} \mathscr{A}'$. Then $\hat{E}$ is a biextension of $(\hat{A}, \hat{A}')$ by $\hat{\mathbb{G}}_m$ and furthermore $\hat{E}(R_w)$ is a set-theoretic biextension of $(\hat{A}(R_w), \hat{A}'(R_w))$ by $R_w^\times$. For more details about scheme-theoretic and formal biextensions, see [8, Exp. VII] and [14].

Following Mazur and Tate in [10, Sect. 2], when working over a general field F, points in $E(F)$ are determined by triples $[\mathfrak{a}, D, c]$ where (1) $\mathfrak{a} \in I[A(F)]$; (2) $D \in \operatorname{Div}(A)$ satisfies $D \approx 0$, has disjoint support from $\mathfrak{a}$, and maps to $\operatorname{Pic}_F^0(A)$; and (3) $c \in F^\times$. The point $[\mathfrak{a}, D, c] \in E(F)$ so determined lies above the point $(\alpha, \alpha') \in A(F) \times A'(F)$ where α is the sum of $\mathfrak{a}$ and α' is the point defined by the divisor class of D.

DEFINITION 3.1. Let Y be an abelian group and suppose $\rho: F^\times \to Y$ is a homomorphism. A *ρ-splitting* is a map

$$\psi: E(F) \to Y,$$

which satisfies for all $[\mathfrak{a}, D, c] \in E(F)$,

(a) $\psi([\mathfrak{a}, D, c]) = \rho(c) \cdot \psi([\mathfrak{a}, D, 1])$ for all $c \in F^\times$;
(b) $\psi([\mathfrak{a}, D, 1])$ is bimultiplicative in $\mathfrak{a}$ and D;
(c) $\psi([\mathfrak{a}, D, 1]) = \rho(f(\mathfrak{a}))$ if $D = \operatorname{div}(f)$;
(d) $\psi([\mathfrak{a}_\alpha, D_\alpha, 1]) = \psi([\mathfrak{a}, D, 1])$ for $\alpha \in A(F)$.

Remark 3.2. We write Y with a multiplicative group law instead of an additive one. The theta functions in Section 5 have natural multiplicative relations, and in Section 6 we will be considering canonical heights which have values in the multiplicative groups of certain fields. In most situations the value group of a canonical height is an additive group, e.g., $\mathbb{R}$ in the case of the Néron–Tate height and $\mathbb{Q}_p$ in the case of p-adic heights.

When A is defined over a local field K_w, Mazur and Tate provide many general situations where we can choose canonical ρ-splittings. We discuss some particular cases, important for our purposes, and so suppose that K_w is a nonarchimedean local field and that Y is uniquely divisible. For the cases where K_w is archimedean or Y is not uniquely divisible, see [10].

EXAMPLE 3.3. If ρ is unramified ($\rho(R_w^\times) = 1$), then $\psi := \psi_{\mathrm{MT}}$ is the unique ρ-splitting such that $\psi(\mathscr{E}(R_w)) = 1$. E.g., if $\rho = -\log|\cdot|_{K_w}$, then

$$\psi([\mathfrak{a}, D, 1]) = (D, \mathfrak{a})_{K_w},$$

where $(D, \mathfrak{a})_{K_w}$ is the Néron symbol (see [15, Sect. II.9]). In this case ψ takes on values in $\mathbb{Q}$ with denominators bounded by the exponent of $\mathscr{A}_0(\mathbb{F}_w)/\mathscr{A}_0^0(\mathbb{F}_w)$. In general, we have

$$\psi([\mathfrak{a}, D, 1]) = \rho(\pi)^{(D, \mathfrak{a})_{K_w}}.$$

EXAMPLE 3.4. If A has semistable ordinary reduction, then $\hat{E}$ is trivial (see [10, Sects. 1.9, 5.11.5] or [14, Sect. 5]) and has a unique splitting $\psi_0: \hat{E} \to \hat{\mathbb{G}}_m$. Then $\psi := \psi_{\mathrm{MT}}$ is the unique ρ-splitting such that $\psi|_{\hat{E}(R_w)} = \rho \circ \psi_0$. We will investigate this case in more detail in Section 5.

If we should consider (3.3) and (3.4) simultaneously, the resulting ρ-splitting is the same.

The canonical ρ-splitting ψ_{MT} satisfies many functorial properties with respect to base extensions, isogenies, norms, etc.; see [10, Sect. 1.10]. In particular, if L_w/K_w is a finite separable extension and $\rho_{L_w}\colon L_w^\times \to Y$ is defined by

$$\rho_{L_w}(a) := \rho(\mathbf{N}_{K_w}^{L_w}(a))^{1/[L_w : K_w]}, \tag{2}$$

then for all $x \in E(K_w)$,

$$\psi_{\mathrm{MT},\, K_w}(x) = \psi_{\mathrm{MT},\, L_w}(x), \tag{3}$$

where $\psi_{\mathrm{MT},\, K_w}$ and $\psi_{\mathrm{MT},\, L_w}$ are the Mazur–Tate canonical splittings for ρ and ρ_{L_w}, respectively.

4. CANONICAL HEIGHTS VIA BIEXTENSIONS

We now review the canonical height constructions of Mazur and Tate (see [10, Sects. 3–4]). Suppose that K is a global field, and let $\mathbf{A}_K^\times$ denote its idele group. Assume that Y is a uniquely divisible abelian group, and suppose we are given a homomorphism

$$\rho = (\rho_w)\colon \mathbf{A}_K^\times \to Y,$$

such that (1) $\rho_w = 1$ if w is archimedean; (2) $\rho_w(R_w^\times) = 1$ for all but finitely many w; and (3) for all $c \in K^\times$ we have $\prod \rho_w(c) = 1$. Mazur's and Tate's constructions are a bit more general than is necessary for the specific heights we will examine; therefore, for simplicity we will at times take on some additional hypotheses. For full details see [10].

Let A be an abelian variety defined over K such that A has semistable ordinary reduction at all places w for which $\rho_w(R_w^\times) \neq 1$. By the discussion in Section 3 there is a canonical ρ_w-splitting $\psi_w\colon E(K_w) \to Y$ for each finite place w. In this case ρ is said to be *admissible*. The following proposition defines a global canonical height and also provides its local decomposition.

PROPOSITION 4.1 (Mazur–Tate [10, Lemma 3.1]). *Suppose A/K is an abelian variety and $\rho\colon \mathbf{A}_K^\times \to Y$ is admissible. Then there is a unique pairing*

$$\langle\, ,\rangle_\rho\colon A(K) \times A'(K) \to Y$$

defined so that if $x \in E(K)$ lies above $(\alpha, \alpha') \in A(K) \times A'(K)$, then

$$\langle \alpha, \alpha' \rangle_\rho = \prod_{w \text{ discrete}} \psi_w(x_w),$$

where $x_w \in E(K_w)$ is obtained from x via the inclusion $K \subset K_w$.

Because of the corresponding properties for local splittings, this canonical height also satisfies many functorial properties with respect to base-change, isogeny, norms, etc. (see [10, Sect. 3.4]). Suppose that A is defined over K and that $\rho_K\colon \mathbf{A}_K^\times \to Y$ is admissible. If L/K is a finite separable extension, then the map $\rho_L\colon \mathbf{A}_L^\times \to Y$ defined by

$$\rho_L((a_w)_w) = \rho_K(\mathbf{N}_K^L(a_w)_w)^{1/[L:K]}, \qquad a \in \mathbf{A}_L^\times, \tag{4}$$

is also admissible. Here $\mathbf{N}_K^L\colon \mathbf{A}_L^\times \to \mathbf{A}_K^\times$ is the norm map on ideles compatible with the norm map $\mathbf{N}_K^L\colon L^\times \to K^\times$. Since $\rho_L|_{\mathbf{A}_K^\times} = \rho_K$, we can thus define an absolute height

$$\langle\, ,\rangle_{\rho_K}\colon A(K^{\mathrm{sep}}) \times A'(K^{\mathrm{sep}}) \to Y. \tag{5}$$

From this definition we see that $\langle\, ,\rangle_{\rho_K}$ is also $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant.

Furthermore, if $\phi\colon A \to B$ is an isogeny and $\phi'\colon B' \to A'$ is its dual, then

$$\langle \phi(\alpha), \beta' \rangle_{\rho_K} = \langle \alpha, \phi'(\beta') \rangle_{\rho_K}, \tag{6}$$

where the left-hand side is the Mazur–Tate pairing on $B \times B'$ and the right-hand side is the pairing on $A \times A'$.

Example 4.2 (Global p-adic heights). Suppose A is defined over a number field K and has good ordinary reduction at every place of K above p. Every $\mathbb{Z}_p$-extension L/K which is unramified away from places $v\,|\,p$ induces a continuous homomorphism via Artin reciprocity,

$$\rho_{L/K}\colon \mathbf{A}_K^\times \to \mathbb{Q}_p.$$

Because of the ordinary reduction assumption on A, it follows that $\rho_{L/K}$ is admissible, and Mazur and Tate show [10, Sect. 4.4] that the resulting canonical pairing is the same as Schneider's p-adic height pairing in [19].

Example 4.3 (Global function fields: v-adic and ∞-adic heights). Let $k = \mathbb{F}_q(t)$ and $R = \mathbb{F}_q[t]$. We fix a place v of k and a uniformizer π_v in $\mathbb{F}_v k$, which if v is finite, generates a prime ideal in $\mathbb{F}_v R$. If $v = \infty$, we take $\pi_\infty = 1/t$. Let $\mathbb{C}_v$ be the completion of the algebraic closure of k_v. We note that the groups $U^1(\mathbb{C}_v)$ (the 1-units of $\mathbb{C}_v$) and $\pi_v^{\mathbb{Q}} \cdot U^1(\mathbb{C}_v)$ are uniquely divisible. Each $x \in \mathbb{C}_v^\times$ can be written uniquely as $x = \zeta \pi_v^r u$, with $\zeta \in \overline{\mathbb{F}}_v$, $r \in \mathbb{Q}$, and $u \in U^1(\mathbb{C}_v)$. We let $\langle x \rangle = \langle x \rangle_v = \pi_v^r u \in \pi_v^{\mathbb{Q}} \cdot U^1(\mathbb{C}_v)$ be the *positive part* of x with respect to π_v. We take

$$Y = \begin{cases} t^{\mathbb{Q}} \cdot U^1(\mathbb{C}_\infty) & \text{if} \quad v = \infty, \\ U^1(\mathbb{C}_v) & \text{otherwise,} \end{cases}$$

and define $\rho_k^{(v)}\colon \mathbf{A}_k^\times \to Y$ by a product over the places $\bar{v}$ of k,

$$\rho_k^{(v)}((a_{\bar{v}})_{\bar{v}}) := \frac{1}{\langle a_v \rangle} \prod_{\bar{v} \neq \infty} \pi_{\bar{v}}^{\mathrm{ord}_{\bar{v}}(a_{\bar{v}})},$$

where $\pi_{\bar{v}}$ is the positive part in $\mathbb{F}_v R$ of a generator of the ideal corresponding to $\bar{v}$.

If A is defined over a global function field K/k, and A has semistable ordinary reduction at each place of K above v, then $\rho_K^{(v)}\colon \mathbf{A}_K^\times \to Y$ as defined in (4) is admissible. Thus Proposition 4.1 defines a canonical height pairing

$$\langle\, ,\, \rangle_v\colon A(K^{\mathrm{sep}}) \times A'(K^{\mathrm{sep}}) \to \mathbb{C}_v^\times,$$

which we refer to as the *v-adic height* or *∞-adic height* depending on whether v is finite or infinite. For $w \mid \bar{v}$ and $\bar{v} \neq v$ finite, we note that the $\rho_{K_w}^{(v)}$-splitting ψ_w is given by

$$\psi_w([\mathfrak{a}, D, 1]) = \pi_{\bar{v}}^{f_w (D,\, \mathfrak{a})_w/[K:k]}, \tag{7}$$

where $f_w = [\mathbb{F}_w : \mathbb{F}_q]$ and $(D, \mathfrak{a})_w$ is the Néron symbol at w.

5. THETA FUNCTIONS AND LOCAL HEIGHTS

In this section we investigate the connection between the local heights defined in Section 4 via canonical ρ-splittings and suitably defined nonarchimedean theta functions. Under certain conditions, we can explicitly construct these theta functions as limits of certain power series on the formal group of the abelian variety. These are essentially the same nonarchimedean theta functions studied in other contexts by Barsotti [1], Breen [3], Candilera and Cristante [4–6] and Norman [16, 17]. We pay special attention to the field of definition. Moreover, we follow the construction of Norman [16] and also of the Mazur–Tate sigma function [11]. We show in Proposition 5.5 that in the elliptic curve case these theta functions are in fact the Mazur–Tate sigma function.

Throughout this section we let A be an abelian variety defined over a local field K_w which has semistable ordinary reduction. We assume that the residue characteristic is $p \geqslant 3$. For a homomorphism

$$\rho\colon K_w^\times \to Y,$$

our goal is to describe explicitly the canonical ρ-splitting $\psi_{\mathrm{MT}}\colon E(K_w) \to Y$, as defined in (3.4).

Assumption 5.1. We will assume that the abelian variety A has a totally symmetric ample K_w-rational divisor X such that (a) X is effective, and (b) the polarization $\phi_X\colon A \to A'$ defined by $\alpha \mapsto$ (class of $X - \tau_\alpha^* X$) has degree prime to the residue characteristic of K_w. Here *totally symmetric* is defined in the sense of Mumford [13], where in particular $[-1]^* X = X$ and for any divisor D, $[-1]^* D + D$ is totally symmetric. Note also by (b) that ϕ_X induces an isomorphism between $\hat{A}$ and $\hat{A}'$.

Remark 5.2. The conditions on A in (5.1) are automatically satisfied by the Jacobian of a curve defined over K_w with a K_w-rational point. We take $X = \Theta + [-1]^* \Theta$ where Θ is the theta divisor (see [12, Sect. 6]). Because $[-1]^* \Theta$ is a K_w-translate of Θ, the degree of ϕ_X is 2^{2d} and prime to p, since $p \geqslant 3$.

We begin with some standard observations and definitions. For any n, let $\hat{A}[n]$ denote the kernel of $[n]\colon \hat{A} \to \hat{A}$. There is a chain of isogenies $b_{k,n}$ of abelian varieties defined over K_w,

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_n \to \cdots,$$

so that for $0 \leqslant k \leqslant n$ the kernel of $b_{k,n}\colon A_k \to A_n$ is $\hat{A}_k[p^{n-k}]$. Because $\hat{A}[p^n]$ is a finite flat subgroup scheme over R_w of $\mathscr{A}$, each A_n is an abelian variety defined over K_w. We then construct isogenies $a_{n,k}\colon A_n \to A_k$ such that $a_{n,k} \circ b_{k,n} = [p^{n-k}]_{A_k}$ and $b_{k,n} \circ a_{n,k} = [p^{n-k}]_{A_n}$. We take $b_n = b_{0,n}$ and $a_n = a_{n,0}$. Furthermore, since A has ordinary reduction, the induced maps

$$a_{n,k}\colon \hat{A}_n \simeq \hat{A}_k \tag{8}$$

are isomorphisms of formal groups.

Let $X \in \mathrm{Div}_{K_w}(A)$ be a totally symmetric ample divisor. For any $n \geqslant 0$ let $X_n := a_n^* X$. The polarization maps $\phi_X\colon A \to A'$ and $\phi_{X_n}\colon A_n \to A_n'$ fit into the commutative diagram

$$\begin{tikzcd} A_n \arrow[r, "\phi_{X_n}"] \arrow[d, shift right, "b_n"'] & A_n' \arrow[d, shift right, "b_n'"'] \\ A \arrow[r, "\phi_X"'] \arrow[u, shift right, "a_n"'] & A' \arrow[u, shift right, "a_n'"'] \end{tikzcd} \tag{9}$$

where a_n' and b_n' are the dual maps.

For $\ell, m > 0$ we let $R_\ell := R_w/(\pi)^\ell$ and $R_{\ell,m} := R_\ell[z_1, \ldots, z_d]/(z_1, \ldots, z_d)^m$. Let $R_w[\![z]\!]$ be the power series ring $R_w[\![z_1, \ldots, z_d]\!]$ and L be its quotient field. If we abbreviate $\mathscr{A} \times R_{\ell,m} = \mathscr{A} \times_{R_w} R_{\ell,m}$ for the base extension to $R_{\ell,m}$ of $\mathscr{A}$, then for a divisor $D \in \mathrm{Div}(A \times_{K_w} L)$, we say that

$$D \sim 0 \qquad (\bmod\ \pi^\ell, \deg m), \tag{10}$$

if the pullback of D under $\mathscr{A} \times R_{\ell,m} \to \mathscr{A} \times R_w[\![z]\!]$ is linearly equivalent to 0. If D satisfies the condition in (10), then there is a rational function $f \in L(A)$ such that the pullback of $\operatorname{div}(f)$ to $\mathscr{A} \times R_{\ell,m}$ is D, for which we write

$$\operatorname{div}(f) \equiv D \qquad (\operatorname{mod} \pi^\ell, \deg m).$$

For α and β two $R_w[\![z]\!]$-valued points of $\mathscr{A} \times R_w[\![z]\!]$, we will say

$$\alpha \equiv \beta \qquad (\operatorname{mod} \pi^\ell, \deg m),$$

if their specializations to $\mathscr{A} \times R_{\ell,m}$ are the same.

If ε is an $R_{\ell,m}$-valued point of $\mathscr{A} \times R_{\ell,m}$ whose specialization to $\mathscr{A} \times \mathbb{F}_w$ is the identity, then via the isomorphism in (8), we define $\varepsilon_n := a_n^{-1}(\varepsilon)$ in $\mathscr{A}_n \times R_{\ell,m}$. For n sufficiently large (say $n \geqslant \ell + \log_p(m)$), we have $[p^n](\varepsilon) = O$, and hence ε is in the kernel of b_n on $\mathscr{A} \times R_{\ell,m}$. Therefore, by considering the various maps in (9), it follows that $a_n' \phi_X(\varepsilon) = O$ in $\mathscr{A}_n' \times R_{\ell,m}$. Thus by composition,

$$\phi_{X_n}(\varepsilon_n) \equiv 0 \qquad (\operatorname{mod} \pi^\ell, \deg m). \tag{11}$$

From this it follows that

$$X_n - \tau_{\varepsilon_n}^* X_n \sim 0 \qquad (\operatorname{mod} \pi^\ell, \deg m), \tag{12}$$

for n sufficiently large.

Now choosing formal uniformizing parameters $z_1, \ldots, z_d$ on A at O selects a coherent sequence of points

$$\gamma_{\ell,m} \colon \operatorname{Spec} R_{\ell,m} \to \mathscr{A} \times R_{\ell,m},$$

such that the image of the closed point of $\operatorname{Spec} R_{\ell,m}$ is the identity on the special fiber of $\mathscr{A} \times R_{\ell,m}$. The completion of $\mathscr{O}_{\mathscr{A},O}$ with respect to $\mathfrak{m}_{\mathscr{A},O} = (z_1, \ldots, z_d, \pi)$ is $R_w[\![z]\!]$. Thus, $\hat{A}$ is the formal spectrum of $R_w[\![z]\!]$, and we can take L for the field of rational functions on $\hat{A}$. Since the quotient field of $\mathscr{O}_{\mathscr{A},O}$ is $K_w(A)$, $K_w(A)$ is naturally a subfield of L. In this way, we produce a tower of function fields

$$K_w(A) = K_w(A_0) \subset K_w(A_1) \subset \cdots \subset L,$$

where for $f \in K_w(A_n)$, we take $f \mapsto \hat{f} = f \circ a_n^{-1}$ (as power series in z).

Following Norman [16, Sections I.3–4], we construct $\Theta_X(z) \in R_w[\![z]\!]$. Let $\gamma \colon \operatorname{Spec} R_w[\![z]\!] \to \mathscr{A} \times R_w[\![z]\!]$ be the limit of $\gamma_{\ell,m}$, let $\gamma_n := a_n^{-1}(\gamma)$ by (8), and likewise $\gamma_{\ell,m,n} := a_n^{-1}(\gamma_{\ell,m})$. Let $\varepsilon := [2]^{-1}(\gamma)$ and $\varepsilon_n := [2]^{-1}(\gamma_n)$, where multiplication by 2 is invertible on the formal group $\hat{A}$ since $p \geqslant 3$. By (12), if n is sufficiently large, there is a rational function $F_{\ell,m,n} \in L(A_n)$ such that

$\operatorname{div}(F_{\ell,m,n}) \equiv X_n - \tau_{\varepsilon_n}^* X_n \pmod{\pi^\ell, \deg m}$. This function $F_{\ell,m,n}$ is fixed so that as a function on A_n,

$$\tau_{-\varepsilon_n}^* F_{\ell,m,n} \cdot [-1]^* F_{\ell,m,n} \equiv 1 \qquad (\operatorname{mod} \pi^\ell, \deg m). \tag{13}$$

For $k \leqslant n$ both sufficiently large, one can take $F_{\ell,m,n} = a_{n,k}^* F_{\ell,m,k}$, which induces on $\hat{A}(R_w[\![z]\!])$ an equality $\hat{F}_{\ell,m,n} = \hat{F}_{\ell,m,k}$. In this case, it makes sense to take $\hat{F}_{\ell,m} := \hat{F}_{\ell,m,n}$.

DEFINITION 5.3. Since X is totally symmetric and defined over K_w, as a Cartier divisor $X = \{(U_i, f_i)\}$ such that $X|_{U_i} = \operatorname{div}(f_i)|_{U_i}$, $[-1]^* U_i = U_i$, and $[-1]^* f_i = f_i$, where U_i and f_i are defined over K_w. Let

$$\Theta_{X,\ell,m}(z) := \hat{F}_{\ell,m}(\delta+\varepsilon)^{-1} \hat{F}_{\ell,m}(\delta)^{-1} f_i(\delta)|_{\delta=O} \in R_{\ell,m}. \tag{14}$$

If $\ell' \geqslant \ell$ and $m' \geqslant m$, one sees that $F_{\ell',m',n} \equiv F_{\ell,m,n} \pmod{\pi^\ell, \deg m}$ for n sufficiently large, and so the limit

$$\Theta_X(z) := \lim_{\ell,m\to\infty} \Theta_{X,\ell,m}(z)$$

exists and can be specified to lie in $K_w \cdot R_w[\![z]\!]$.

If $\alpha \in \hat{A}(\bar{R}_w)$, then we define $\Theta_{X_\alpha}(z) := \Theta_X(z-\alpha)$. Suppose $D \approx 0$ is a K_w-rational divisor on A, and $D \sim \sum_\alpha m_\alpha X_\alpha$, where α runs over points in $\hat{A}(\bar{R}_w)$. We define

$$\Theta_D(z) := f(z) \prod_\alpha \Theta_{X_\alpha}(z)^{m_\alpha} \in L, \tag{15}$$

where $\operatorname{div}(f) = D - \sum m_\alpha X_\alpha$.

In all the cases above, $\Theta_X(z)$ and $\Theta_D(z)$ are defined only up to a constant from $K_w^\times$. Also $\Theta_D(z)$ depends on the choices of X and the representation $D \sim \sum m_\alpha X_\alpha$. We analyze these dependencies in the next theorem.

If $\mathfrak{a} = \sum n_\beta(\beta) \in I[\hat{A}(\bar{R}_w)]$, then for $D \approx 0$ as in the previous paragraph, we let $\Theta_D(\mathfrak{a}) := \prod \Theta_D(\beta)^{n_\beta}$.

THEOREM 5.4. *Suppose A has an ample totally symmetric K_w-rational divisor X satisfying the properties in (5.1). Let $\mathfrak{a} \in I[\hat{A}(\bar{R}_w)]$ and $D \in \operatorname{Div}(A)$ have disjoint supports, with $D \approx 0$ representing a point in $\hat{A}'(R_w)$.*

(a) *If $\mathfrak{a}$ is defined over K_w, then $\Theta_D(\mathfrak{a}) \in K_w^\times$.*

(b) *$\Theta_D(\mathfrak{a})$ is independent of the choices of uniformizing parameters for $\hat{A}$, the divisor X, and the representation of $D \sim \sum m_\alpha X_\alpha$.*

(c) *$\Theta_D(\mathfrak{a})$ is bimultiplicative in $\mathfrak{a}$ and D.*

(d) *If* $D \sim 0$*, then* $\Theta_D(\mathfrak{a}) = f(\mathfrak{a})$ *where* $D = \operatorname{div}(f)$.

(e) $\Theta_{D_\alpha}(\mathfrak{a}_\alpha) = \Theta_D(\mathfrak{a})$ *for any* $\alpha \in \hat{A}(\bar{R}_w)$.

Proof. The assumption in (5.1) implies that $D \sim X - X_{-\alpha}$ for $\alpha \in \hat{A}(R_w)$. Thus $\Theta_D(z)$ is defined and $\Theta_D(z) \in L$. Because $\mathfrak{a}$ is disjoint from D, (14) implies that $\Theta_D(\mathfrak{a}) \neq 0$. These prove (a).

For part (b), first for a given X, $\Theta_X(z)$ does not depend on the choice of uniformizing parameters for $\hat{A}$. Indeed the specialization to a specific point in $\hat{A}(R_w)$ of the function $F_{\ell,m,n}$ as defined in (13) is the same, regardless of the choices of uniformizing parameters.

Now suppose $D \sim \sum m_\alpha X_\alpha \sim \sum n_\beta Y_\beta$ are two different representations of D, for totally symmetric divisors $X, Y \in \operatorname{Div}_{K_w}(A)$ satisfying (5.1), and suppose $\operatorname{div}(f) = D - \sum m_\alpha X_\alpha$ and $\operatorname{div}(g) = D - \sum n_\beta Y_\beta$ as in (15). For $\alpha \in \hat{A}(\bar{R}_w)$ let $\hat{F}_{X,\alpha}(\delta)$ be defined on $\hat{A}$ as in (13) and (14) so that

$$\hat{F}_{X,\alpha}(\delta) = \hat{F}_{\alpha,\ell,m,n}(\delta + \varepsilon - [2]^{-1}(\alpha))^{-1}\, \hat{F}_{\alpha,\ell,m,n}(\delta)^{-1}\, f_i(\delta),$$

with $\operatorname{div}(F_{\alpha,\ell,m,n}) \equiv X_n - \tau^*_{\varepsilon_n - [2]^{-1}(\alpha_n)} X_n$ for n sufficiently large. Similarly define $\hat{G}_{Y,\beta}$. Then by (14),

$$\prod_{\alpha,\beta} \frac{\Theta_{X_\alpha,\ell,m}(z)^{m_\alpha}}{\Theta_{Y_\beta,\ell,m}(z)^{n_\beta}} \equiv \prod_{\alpha,\beta} \frac{\hat{F}_{X,\alpha}(\delta)^{m_\alpha}}{\hat{G}_{Y,\beta}(\delta)^{n_\beta}}\bigg|_{\delta = O} \qquad (\operatorname{mod} \pi^\ell, \deg m).$$

The right-hand side is the evaluation at O of the rational function in $L(A)$ with divisor $\tau^*_\gamma(\sum m_\alpha X_\alpha - \sum n_\beta Y_\beta) = \tau^*_\gamma \operatorname{div}(g/f)$. Thus $\Theta_D(z)$ is independent of the use of X or Y and the representation of D in terms of them. Moreover, because $\mathfrak{a} \in I[\hat{A}(\bar{R}_w)]$, $\Theta_D(\mathfrak{a})$ is independent of the choices of the constant multiples of rational functions in (13)–(15).

For linearity in $\mathfrak{a}$, part (c) follows from the definition of $\Theta_D(\mathfrak{a})$. The linearity in D follows from part (a): if we write $D = D_1 + D_2$, then the value of $\Theta_D(\mathfrak{a})$ does not depend on the representation of D, D_1, or D_2 in terms of X, and so in particular $\Theta_D(\mathfrak{a}) = \Theta_{D_1}(\mathfrak{a})\,\Theta_{D_2}(\mathfrak{a})$. Part (d) follows from (15) and part (a). Part (e) follows from the definition that $\Theta_{X_\alpha}(z) = \Theta_X(z - \alpha)$. ∎

In the context of Section 3, these theta functions provide canonical ρ-splittings. In the case of (3.4), we see that if A has semistable ordinary reduction, then $\psi_0 \colon \hat{E} \to \hat{\mathbb{G}}_m$ defined by

$$\psi_0([\mathfrak{a}, D, 1]) := \Theta_D(\mathfrak{a}),$$

for $[\mathfrak{a}, D, 1] \in \hat{E}(\bar{R}_w)$ is the unique splitting of the formal biextension $\hat{E}$ in the following way. By Theorem 5.4 the restriction of ψ_0 to the inverse image in $\hat{E}(R_w)$ of a point in either $\hat{A}(R_w)$ or $\hat{A}'(R_w)$ is a group homomorphism. Thus we need to show that ψ_0 is a formal morphism. Given formal parameters z on $\hat{A}$, then for the effective divisor X in (5.1) it follows that coordinate functions for $\phi_X(z) = (\text{class of } X - \tau_z^* X)$ provide formal parameters on $\hat{A}'$. If $\alpha \in \hat{A}(\bar{R}_w)$ and α, $z+\alpha$ are disjoint from $X - \tau_{z'}^* X$ for $z' \in \hat{A}(\bar{R}_w)$, then

$$\psi_0([(z+\alpha)-(\alpha), X-\tau_{z'}^* X, 1]) = \frac{\Theta_X(z+\alpha)\,\Theta_X(z'+\alpha)}{\Theta_X(\alpha)\,\Theta_X(z+z'+\alpha)}.$$

By the definition of Θ_X, this can be expressed as a power series in z and z' with coefficients in $\bar{R}_w$ and bounded denominators. By additional applications of Theorem 5.4, every $\psi_0([\mathfrak{a}, D, 1])$ can be expressed as a power series in z and z', and thus ψ_0 gives a formal morphism.

If $\rho\colon K_w^\times \to Y$ is a homomorphism, then the canonical ρ-splitting of Mazur and Tate is the ρ-splitting $\psi\colon E(K_w) \to Y$ such that $\psi|_{\hat{E}(R_w)} = \rho \circ \psi_0$.

In the case that A is an elliptic curve, these theta functions can also be directly compared to the Mazur–Tate sigma function, as defined in [11].

PROPOSITION 5.5. *Let A be an elliptic curve defined over K_w with semistable ordinary redaction. Let $X = 2(O)$. Then*

$$\Theta_X(z) = C\sigma_A(z)^2,$$

where σ_A is a Mazur–Tate sigma function and $C \in K_w^\times$ is a constant.

Proof. Let $X = 2(O)$, and let $f_n \in K_w(A_n)$ have divisor $X_n - 2p^n(O_n) = 2a_n^*(O) - 2p^n(O_n)$, where O_n is the identity element of A_n. It follows that

$$X_n - \tau_\beta^* X_n = \operatorname{div}(f_n) - \tau_\beta^* \operatorname{div}(f_n) - 2p^n((-\beta)-(O_n)),$$

for any point $\beta \in A_n(K_w)$.

Let z be a uniformizing parameter at O for $\hat{A}/R_w$ such that $z \circ [-1] = -z$, and let $f \in K_w(A)$ be an even function, regular on $\hat{A}$, which defines the divisor X in an open neighborhood of O containing $\hat{A}$. Let $z_n = a_n^{-1}(z)$. Fix $\ell, m > 0$. There is a $n_0 > 0$ so that $[p^{n_0}](z) \equiv O \pmod{\pi^\ell, \deg m}$. For any $n \geqslant n_0$, there is a function $g_n \in L(A_n)$ so that

$$\operatorname{div}(g_n) = p^{n_0}([2]^{-1}(-z_n)) - (p^{n_0}-1)(O_n) - ([p^{n_0}]([2]^{-1}(-z_n))).$$

Then we can pick $F_{\ell,m,n} \in L(A_n)$ to be (with $\delta_n = a_n^{-1}(\delta)$)

$$F_{\ell,m,n}(\delta_n) = \frac{f_n(\delta_n)}{f_n(\delta_n + [2]^{-1}(z_n))\,g_n(\delta_n)^{2p^{n-n_0}}},$$

and thus $\operatorname{div}(F_{\ell,m,n}) \equiv X_n - \tau_{z_n}^* X_n \pmod{\pi^\ell, \deg m}$. The condition in (13) implies that we further choose g_n so that

$$g_n(\delta_n - z_n)^{2p^{n-n_0}}\, g_n(-\delta_n)^{2p^{n-n_0}} \equiv 1 \qquad (\operatorname{mod} \pi^\ell, \deg m).$$

In the definition of $\Theta_{X,\ell,m}$ in (14), we see that

$$\begin{aligned}\Theta_{X,\ell,m}(z) &= \frac{\hat{f}_n(\delta+z)(\hat{g}_n(\delta+[2]^{-1}(z))\,\hat{g}_n(\delta))^{2p^{n-n_0}}}{\hat{f}_n(\delta)} \cdot f(\delta)\bigg|_{\delta=O}\\ &= z^{2p^n}\hat{f}_n(z)\left(\frac{(\hat{g}_n(\delta+[2]^{-1}(z))\,\hat{g}_n(\delta))^{2p^{n-n_0}}\, f(\delta)}{\hat{f}_n(\delta)(\delta+z)^{2p^n}}\right)\bigg|_{\delta=O} \in R_{\ell,m}.\end{aligned}$$

By considering the divisor restricted to $\hat{A}$ of the function in δ within the parentheses above, and after substituting in $\delta = O$, we find that

$$\Theta_{X,\ell,m}(z) \equiv Cz^{2p^n}\hat{f}_n(z)\, u(z)^{p^{n-n_0}} \qquad (\operatorname{mod} \pi^\ell, \deg m),$$

where $C \in K_w^\times$ is a constant independent of z, and $u(z)$ is a 1-unit in $R_w[\![z]\!]$. As $n \to \infty$, then $u(z)^{p^{n-n_0}} \to 1$, and by comparison with the construction in [11], we confirm that as $\ell, m \to \infty$,

$$\Theta_{2(O)}(z) = C\sigma_A(z)^2,$$

for some constant $C \in K_w^\times$. ∎

6. UNIVERSAL NORMS

In this section we restrict our attention to abelian varieties defined over global function fields in positive characteristic. In particular we examine the v-adic heights studied in Example 4.3.

We pick up with the notation of (4.3). We let $k = \mathbb{F}_q(t)$ and $R = \mathbb{F}_q[t]$ and assume $p \geqslant 3$. Given a polynomial a in R, we let $C[a]$ denote the a-torsion on the Carlitz module. It is well known that $C[a] \cong R/(a)$ as an R-module, and the field $k(C[a])$ is an abelian extension of k with Galois group naturally isomorphic to $R/(a)^*$. Indeed, if we let $\lambda \in C[a]$ be a generator for $C[a]$ as an R-module, then for $b \in R$,

$$(b, k(C[a])/k)(\lambda) = C_b(\lambda), \tag{16}$$

where the left-hand side is the Artin symbol and the right-hand side is λ multiplied by b on the Carlitz module. For more details on the Carlitz module and Drinfeld modules in general, see Goss [7, Chaps. 4, 7].

We now fix a finite place v of k. We let k^n be the subfield of $k(C[\pi_v^n])$ fixed by the subgroup $\mathbb{F}_v^\times \subset \mathrm{Gal}(k(C[\pi_v^n])/k)$. The field $k^\infty := \bigcup k^n$ is an abelian pro-p-extension of k which is totally ramified at v, unramified away from v, and totally split at ∞ (see [7, Sect. 7.5]). Notably there is an isomorphism via (16),

$$\Phi\colon \mathrm{Gal}(k^\infty/k) \simeq U^1(k_v). \tag{17}$$

If $\rho_k^{(v)}\colon \mathbf{A}_k^\times \to U^1(k_v)$ is the map defined in (4.3), then it follows from the definition of the Artin map that

$$\rho_k^{(v)} = \Phi \circ \mathrm{rec}\colon \mathbf{A}_k^\times \to U^1(k_v), \tag{18}$$

where rec is the Artin reciprocity map on $\mathbf{A}_k^\times$.

If K/k is a finite separable extension, then we let $K^n := Kk^n$. Then $\mathrm{Gal}(K^\infty/K)$ can be identified with a subgroup of $\mathrm{Gal}(k^\infty/k)$. It follows that

$$(\rho_K^{(v)})^{[K:k]} = \Phi|_{\mathrm{Gal}(K^\infty/K)} \circ \mathrm{rec}\colon \mathbf{A}_K^\times \to U^1(k_v), \tag{19}$$

where $\rho_K^{(v)}$ is defined in (4.3).

Given an abelian variety A defined over K which has good ordinary reduction at each place of K above v, we let $\langle\ ,\ \rangle_v$ denote the v-adic height pairing. We let $A(L)_p := A(L) \otimes \mathbb{Z}_p$ for any L/K. We let the *universal norms* of $A(K)$ be the subgroup $U_v(A(K)) \subset A(K)_p$ defined by

$$U_v(A(K)) := \bigcap \mathbf{N}_K^{K^n}(A(K^n)_p).$$

Since $U^1(k_v)$ is a $\mathbb{Z}_p$-module, by linearity we can extend the definition of the v-adic pairing to

$$\langle\ ,\ \rangle_v\colon A(K)_p \times A'(K)_p \to U^1(k_v)^{1/(\mu[K:k])} \subset U^1(\mathbb{C}_v), \tag{20}$$

where μ is a common multiple for the exponents of $\mathscr{A}_0(\mathbb{F}_w)/\mathscr{A}_0^0(\mathbb{F}_w)$ for all places $w \nmid v$ and of $\mathscr{A}_0(\mathbb{F}_w)$ for all $w \mid v$.

THEOREM 6.1. *Suppose A/K has good ordinary reduction at every place of K extending v. Let $\varepsilon \in U_v(A(K))$. Then for every $\beta \in A'(K)_p$,*

$$\langle \varepsilon, \beta\rangle_v = 1.$$

Proof. Let $\alpha_n \in A(K^n)_p$ be chosen so that $\mathbf{N}_K^{K^n}(\alpha_n) = \varepsilon$. Then by Galois equivariance,

$$\langle \varepsilon, \beta\rangle_v = \langle \mathbf{N}_K^{K^n}(\alpha_n), \beta\rangle_v = \langle \alpha_n, \beta\rangle_v^{[K^n:K]}. \tag{21}$$

The exponent of $\mathscr{A}_0(\mathbb{F}_w)/\mathscr{A}_0^0(\mathbb{F}_w)$ is unchanged in an extension where w is unramified and the exponent of $\mathscr{A}_0(\mathbb{F}_w)$ is unchanged in an extension where w is ramified and a place of good reduction, in particular in K^n/K. Therefore, it follows from (20) that there is a fixed integer μ, independent of n, such that

$$\langle \alpha_n, \beta \rangle_v \in U^1(k_v)^{1/(\mu[K^n:k])}. \tag{22}$$

However, we can say more. For each place w of K^n, the image of the local $\rho_{K^n}^{(v)}$-splitting ψ_w is in $\rho_{K^n}^{(v)}(K_w^n)^{1/\mu}$. By Artin reciprocity and the isomorphism in (16), the subgroup $\mathrm{Gal}(K^\infty/K^n) \subset \mathrm{Gal}(k^\infty/k^n) \subset \mathrm{Gal}(k^\infty/k)$ maps into $U_n^1(k_v) := \{1 + a_n \pi_v^n + a_{n+1}\pi_v^{n+1} + \cdots\}$ under the map in (17), and so by (19)

$$\rho_{K^n}^{(v)}(\mathbf{A}_{K^n}^\times) \subset U_n^1(k_v)^{1/(\mu[K^n:k])}.$$

By (21) we see that in fact

$$\langle \varepsilon, \beta \rangle_v \in U_n^1(k_v)^{1/(\mu[K:k])},$$

for every n. Therefore, $\langle \varepsilon, \beta \rangle_v = 1$. ▮

Remark 6.2. In the situation of Schneider's global p-adic heights in [10, 19], we know that the p-adic height pairing is characterized by its vanishing on the subgroup of universal norms (see [10, Sect. 1.11]). The same can be said for the v-adic height pairing by considering the Mazur–Tate argument as follows.

For each place $w \mid v$, let $K_w^n = K^n \cdot K_w$, and let $\tilde{A}(K_w) = \bigcap_n \mathbf{N}_n A(K_w^n)$, where $\mathbf{N}_n$ is the norm map $\mathbf{N}_{K_w}^{K_w^n}$. Let $E(K_w^n, K_w) \subset E(K_w)$ be the subset of $E(K_w)$ consisting of elements projecting to $A(K_w^n) \times A(K_w)$, and let $\tilde{E}(K_w) = \bigcap_n \mathbf{N}_n E(K_w^n, K_w) \subset E(K_w)$, where $\mathbf{N}_n\colon E(K_w^n, K_w) \to E(K_w)$ is defined by the norm map on the fibers of $E \to A'$.

For each $\beta \in A'(K_w)$, a suitable modification of the main theorem of [9] or Lemma 3 of [19] shows that, for the fiber $\tilde{E}_\beta(K_w)$, the sequence

$$0 \to K_w^\times \to \tilde{E}_\beta(K_w) \to \tilde{A}(K_w) \to 0$$

is exact, continuing under the assumption that A has good ordinary reduction at w. We note that $A(K_w)/\tilde{A}(K_w)$ is not finite but that it does have finite exponent. These facts are sufficient to show that $E(K_w)$ induces on $\tilde{E}(K_w)$ the structure of a biextension of $(\tilde{A}(K_w), A'(K_w))$ by $K_w^\times$. By [10, Sects. 1.5–1.7], there is a unique $\rho_{K,w}^{(v)}$-splitting which vanishes on $\tilde{E}(K_w)$. Theorem 6.1 implies that the Mazur–Tate $\rho_{K,w}^{(v)}$-splitting ψ_w satisfies this property, and so the two splittings must coincide.

When A is an elliptic curve, v-adic heights of this type were studied in [18]. By Proposition 5.5 we observe that the heights defined in that paper coincide exactly with v-adic heights defined here. Moreover, in [18, Corollary 8.8] some hypotheses were obtained for the nondegeneracy of the v-adic height pairing.

In particular, we let $\mathscr{T}_p(A) = \varprojlim \ker V_n$ be the p-adic Tate module for A, where V_n is the dual of the q^nth power Frobenius map on A. We then have the following corollary giving the triviality of the universal norm subgroup in certain cases.

COROLLARY 6.3. *Let A/K be an elliptic curve with good ordinary reduction at both v and ∞. Suppose the natural map* $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(\mathscr{T}_p(A))$ *is surjective and that for all places $w \mid v$ or $w \mid \infty$ of K we have $\mathscr{A}_0(\mathbb{F}_w)[p] = \{O\}$. Then $U_v(A(K)) = \{O\}$.*

The hypothesis above that $\mathscr{A}_0(\mathbb{F}_w)[p]$ is trivial implies in particular that $A(K)[p] = \{O\}$. Thus $A(K)_p$ is torsion free. The corollary then follows directly from Corollary 8.8 in [18] and Theorem 6.1.

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REFERENCES

1. I. Barsotti, Theta functions in positive characteristic, *in* "Journées de géometrie algébrique de Rennes (Rennes 1978), Vol. I," *Astérisque* **63** (1979), 5–16.
2. S. Bloch, A note on height pairings, Tamagawa numbers, and the Birch and Swinnerton–Dyer conjecture, *Invent. Math.* **58** (1980), 65–76.
3. L. Breen, "Fonctions thêta et théorème du cube," Lecture Notes in Mathematics, Vol. 980, Springer-Verlag, Berlin, 1983.
4. M. Candilera and V. Cristante, Bi-extensions associated to divisors on abelian varieties and theta functions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **10** (1983), 437–491.
5. V. Cristante, Theta functions and Barsotti–Tate groups, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **7** (1980), 125–181.
6. V. Cristante, p-adic theta series with integral coefficients, *in* "Cohomologie p-adique," *Astérisque* **119–120** (1984), 169–182.
7. D. Goss, "Basic Structures of Function Field Arithmetic," Springer-Verlag, Berlin, 1996.
8. A. Grothendieck, Groupes de monodromie en géométrie algébrique, *in* "Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (7 I)," Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, Berlin, 1972.
9. J. Lubin and M. I. Rosen, The norm map for ordinary abelian varieties, *J. Algebra* **52** (1978), 236–240.
10. B. Mazur and J. Tate, Canonical height pairings via biextensions, *in* "Arithmetic and Geometry: Papers Dedicated to I. R. Shafarevich on the Occasion of His Sixtieth Birthday" (M. Artin and J. Tate, Eds.), Vol. I, pp. 195–237, Birkhäuser, Boston, 1983.

11. B. Mazur and J. Tate, The p-adic sigma function, *Duke Math. J.* **62** (1991), 663–688.
12. J. S. Milne, Jacobian varieties, *in* "Arithmetic Geometry" (G. Cornell and J. H. Silverman, Eds.), pp. 167–212, Springer-Verlag, New York, 1986.
13. D. Mumford, On the equations defining abelian varieties I, *Invent. Math.* **1** (1966), 287–354.
14. D. Mumford, Bi-extensions of formal groups, *in* "Algebraic Geometry" (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), pp. 307–322, Oxford Univ. Press, London, 1969.
15. A. Néron, Quasi-fonctions et hauteurs sur les variétés abéliennes, *Ann. of Math. (2)* **82** (1965), 249–331.
16. P. Norman, p-adic theta functions, *Amer. J. Math.* **107** (1985), 617–661.
17. P. Norman, Explicit p-adic theta functions, *Invent. Math.* **83** (1986), 41–57.
18. M. A. Papanikolas, Canonical heights on elliptic curves in characteristic p, *Compositio Math.* **122** (2000), 299–313.
19. P. Schneider, p-adic height pairings I, *Invent. Math.* **69** (1982), 401–409.
20. S. Turner, Nonarchimedean theta functions and the Néron symbol, *J. Number Theory* **21** (1985), 1–16.